# A PROCEDURAL ROUTE TOWARD UNDERSTANDING THE CONCEPT OF PROOF

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*In this paper, I describe how undergraduates can develop their understanding of the concept of proof by viewing the act of proving as a procedure. Such undergraduates first understand proof as an* ***algorithm****- or a step-by-step mechanical prescription for proving certain types of statements. The students can then condense this algorithm into a* ***process****- or a shorter list of global, qualitative steps. By reflecting on the process, successful students can view proof as an* ***argument****- something that establishes the veracity of a mathematical statement. I illustrate a student in real analysis who followed this learning progression to learn the concept of proof by induction. Finally, I note that many students who view proof as a process fail to ever view proof as an argument and I discuss the consequences of these students' narrow view of proof.*

## INTRODUCTION

Undergraduates can successfully come to understand advanced mathematical concepts in qualitatively different ways (Pinto and Tall, 1998). Some students develop their understanding of a concept by focusing on the definition of the concept and its logical entailments; others use their intuitive understanding of a concept to give meaning to the concept's definition (Pinto and Tall 1998, 2002). Still other students can learn about a mathematical concept by first viewing it as a process and only later viewing the concept as a static mathematical object. (cf. Sfard, 1991; Tall, Thomas, Davis, Gray, and Simpson, 2000).

The purpose of this paper is to describe one way that undergraduates may come to understand the concept of proof. Proof is an intricate mathematical concept with rigorous, intuitive, procedural, and social dimensions (e.g. Weber, 2002a). As such, Simpson (1995) suggests that there may be more than one way that a student can develop their understanding of this concept. Simpson describes how some students can learn proof logically as rules for manipulation, while others may learn proof as reasoning, with an emphasis on argumentation. In this paper, I suggest another learning progression that students may traverse to understand the concept of proof. Students may develop their understanding of proof by viewing common proof techniques as mechanical procedures. Only later will successful students reason about why they are applying these proof techniques and extract standards for what constitutes a valid proof.

## RESEARCH CONTEXT

The data from this study was collected within the context of a first course in real analysis taught at a regional comprehensive university in the southern United States. This course was taught in a traditional manner by a professor unaffiliated with this study. Six students in the course volunteered to participate in this study. Of these six students, three were taking a proof-oriented mathematics course for the first time. The other three had completed a course entitled "Mathematical Reasoning", a bridge course introducing

students to the concept of proof. However, at the beginning of the real analysis course, these three students all claimed to have little understanding of the concept of proof, and their performance on their first assignments and their early interviews with the author of this paper support these students' self-assessments. Hence all six students in the real analysis course were still developing their conception of proof.

Each students participating in this study met with the author of this paper every other week. There were eight such meetings during the fifteen-week course. During these meetings, students discussed a wide variety of topics with the interviewer, including their attitudes about proof and advanced mathematics. Students also answered questions designed to probe their formal and intuitive understandings of the concepts and proof techniques in the course. Finally, students were asked to construct basic proofs. All interviews were recorded and transcribed.

## A PROCEDURAL APPROACH TOWARD UNDERSTANDING PROOF

In this section, I describe how some undergraduates developed their understanding of proof by first understanding proof as a procedure. Before doing so, let me state that the learning progression I describe below was not the only way that the undergraduates in this study learned the concept of proof. Some undergraduates in this study primarily viewed proofs as the formalization of their intuition (cf. Weber, 2002b). These students used their conception of mathematical proof to produce arguments and used the feedback that they received on assignments and examinations to refine their conception of a mathematical proof. At times, other students viewed constructing proofs as finding strategies for manipulating logical sentences (cf. Weber, 2001). These students' performances will be the subject of future reports.

In this paper, I will concentrate on students who learned about the concept of proof via a procedural route. Such students, when successful, first understood proof as an *algorithm*, then as a *process*, and finally as an *argument*. At this stage, it should be noted that there is much in common between this learning trajectory and the process-object theories of concept acquisition (e.g. Tall et. al., 2000; Dubinsky and McDonald, 2001). I will describe each stage of my hypothesized learning trajectory in more detail below. In the following sub-section, I will present a case study of one students' successful learning of proof by induction within the context of this trajectory.

*Proof as algorithm*- Many of the undergraduates in this study first learned a proof technique (e.g. proof by induction or proving that a sequence converges to a specific value) as a step-by-step prescription for proving certain types of statements. Each of the steps in this algorithm was relatively mechanical and highly tied to a specific type of problem. At this stage, the undergraduates were generally unaware of the overall nature of the procedure that they were incorporating. As a result, undergraduates with an algorithmic understanding of a proof technique often could only apply this technique toward a very specific type of problem. For instance, after many undergraduates first learned how to prove that linear functions were continuous at a particular point, they could not prove similar statements for other types of functions (e.g. quadratic and rational functions). In short, the undergraduates were able to mimic a technique that the professor had demonstrated for them, but they did not really understand what they were doing,

could not generalize the technique, and could not justify why this technique established the veracity of a statement.

*Proof as process*- After the undergraduates applied a proof technique multiple times and in different contexts, some were able to *interiorize* the algorithm into a mathematical process. The algorithm previously viewed as a set of mechanical actions was condensed into a shorter list of global, qualitative steps. These global steps were not highly specified manipulations on a specific type of statement (divide by the leading coefficient of this polynomial), but rather involved accomplishing a general goal (solve an arbitrary inequality, find a delta that satisfies a particular condition). For this reason, the undergraduates who viewed a proof technique as a process were able to divorce the technique from the specific statements it was originally used to prove. This allowed the undergraduates to generalize their proof technique to a larger number of cases, including cases that they may not have seen before. It appeared that seeing the same proof technique applied within different contexts helped undergraduates view a proof technique as a process, as they seemed to extract common global steps from the ostensibly different proof procedures.

Students with a process understanding of a proof method often can flexibly apply this method over a wide range of cases, yet their epistemological understanding of the concept of proof may still be naïve. Many of the undergraduates in this study could apply a proof technique in a large number of cases, but they did not view proving as convincing or explaining, nor did they view a proof as a convincing explanation. Rather such students viewed the act of proving as a process that one applies to receive credit on examinations, not unlike computing a derivative in a first calculus course (cf. Moore, 1994). To these undergraduates, the proof itself is the successful completion of that process. The proofs these undergraduates submitted were not so much arguments, but rather chronological accounts of their work (cf. Dreyfus, 1999). Potential consequences of this narrow view of proof are described in the concluding section.

*Proof as argument*- By reflecting upon the proof procedure that they were applying, successful undergraduates began to think about why they were asked to apply that procedure and what that procedure was designed to accomplish. In this way, some undergraduates were able to extract standards for what constituted a valid proof from the procedure itself. Just as encapsulating processes into mathematical objects is inherently difficult (e.g. Tall et. al., 2000; Dubinsky and McDonald, 2001), it appears that viewing proof as an argument rather than a process was a very challenging transition for the undergraduates to make. Most undergraduates did not even attempt to make sense of the procedures that they were applying.

## A CASE STUDY OF A STUDENT LEARNING PROOF BY INDUCTION

I illustrate the procedural route toward understanding proofs by describing David's progression toward learning proof by induction. The real analysis course was the first proof-oriented course that David enrolled in. Throughout the course, David expressed a strong desire to make sense of his mathematical work and he generally viewed proving as formalizing his intuition. He tried to understand how the concepts of the course fit together, frequently described and reasoned about concepts with descriptive diagrams, and borrowed analysis texts from the library so the he could obtain different viewpoints

on the course topics. In this way, he was not unlike the gifted and successful student described in Pinto and Tall (2002). However, the set theoretic proofs that David was introduced to in the beginning of the semester were so alien to him that he resorted to learning how to write such proofs by rote.

David first learned how to use proof by induction as a linear sequence of steps that he could perform to verify identities involving summations. Five weeks into the course, I asked David to use induction to verify an inequality. When this occurred, he was unable to make much progress beyond establishing the basis case. After I gave David several suggestive hints, he produced what appeared to be a thoughtful proof by induction. The following excerpt reveals that he did not understand what he had just accomplished.

> I: Good.
>
> David: OK, so what do we do now?
>
> I: Well, we uh… do we need to do anything now?
>
> David: Are we done?
>
> I: You tell me. Have we shown the basis case?
>
> David: Yes.
>
> I: Did you show that P(k) implies P(k+1)?
>
> David: Um… I think so.
>
> I: Do we need to do anything more in a proof by induction?
>
> David: I guess not… [pause]…
>
> I: Why did you think that we weren't done before? Do you know why what you wrote down is a correct proof?
>
> David: Well because this proof doesn't look like the other proofs we did [by induction]. The other ones had a lot of equations and this one has sort of just one string. I mean I know it's correct. I see why it is proved for the P(k+2) and P(k+3) case. This one just looked different.

David clearly did not understand the essence of proof by induction. Further, his scheme for validating proofs was at least partially dependent upon how superficially similar his new proof was to previously observed or constructed proofs. On his mid-term examination, David was asked to verify an equality involving products (i.e. Prove $(1 + 1/1)(1+1/2)\ldots(1+1/n) = n+1$). Again, he was unable to make meaningful progress despite claiming to spend 20 minutes on the question. In our interview after the examination, he indicated that he found the question to be unfair, because he'd "done induction using summation, but not induction using multiplication."

The following excerpt taken from our sixth interview twelve weeks into the course illustrates that David had acquired a more flexible understanding of mathematical induction.

> I: Could you describe how you would use induction to prove that $2^{n-1} \leq n!$ ?
>
> David: Well the basis case just gives …. 1 is equal to 1. To solve the inductive step, I would have to see how (n+1)! related to n! and how $2^{n+1}$ related to $2^n$. I think that I would approach it

> in some way of handling the factorial. If I can expand (n+1)! In some way, I can see how it relates to n. If I can see how they are related, I can use my inductive hypothesis.

Although David had trouble articulating his ideas, it appears that he had acquired powerful general strategies for approaching proof by induction. David's use of these global strategies allows him to approach induction problems involving factorials and exponents, even though he had not encountered such problems before. However, as the following excerpt taken from the same interview reveals, David still does not understand why proof by induction is a legitimate proof technique:

> David: And I prove something and I look at it, and I thought, well, you know, it's been proved, but I still don't know that I even agree with it [laughs]. I'm not convinced by my own proof!

By the seventh interview, David had acquired a strong understanding of proof by induction, as the following excerpt demonstrates:

> I: How would you go about proving this by induction [presents the statement: $2^n$ □ n! for $n \geq 4$]? Note that this isn't like regular induction. I'm not asking you to prove it for all natural numbers, just n is bigger than or equal to four.
>
> David: Wouldn't I, wouldn't I just prove it for four and then prove the inductive step?
>
> I: Yes. Yes you would. Could you explain to me why your proof would work?
>
> David: Well because of the domain, I guess, for lack of a better word. You know we start at 4, it works on that first element, then it's going to… if it works there, then it works for the k+1 th element, then it's going to work for five, then six, then seven, and all the bigger elements than that.

In our final interview, I asked David to reflect on how he came to understand induction as a proof technique. By his account, for most of the semester, he understood induction as mimicking a procedure, and he was unsure why the technique showed something was true for all natural numbers. When I asked how he achieved the greater understanding that he had at the end of the course, he responded:

> "I wouldn't say it was any one thing, I guess, practice with it, thinking about what I'm doing, you know talking about it with you and [the professor], just being exposed more to it… I don't know. I think it is a combination of things".

In summary, through the first half of the semester, David understood proof by induction as an algorithm that he could apply to prove identities involving summations. He was not able to apply proof by induction to other statements, such as inequalities or identities involving products; indeed, he thought such questions were unfair. Later, David understood proof by induction as a process. He was able to skillfully approach problems that he had not seen before, but he still did not see why an inductive proof was a convincing mathematical argument. Toward the end of the semester, through an ill-defined combination of exposure, discussion, and reflection, David was able to extract meaning from the process that he was applying to understand why proof by induction was a valid proof technique.

## DISCUSSION AND CONCLUSIONS

The undergraduates in this study often did not view the act of proving as establishing the mathematical certainty of a statement, but rather as a process that one executes to receive credit on assignments and examinations. In this section, I describe how this narrow view

of proof caused these students much difficulty and confusion as the course progressed. When validating their own proofs, these undergraduates would often compare how similar their proof was in form to previous arguments that they had observed or constructed. When validating others' proofs, their judgments were influenced by how similar the observed proof was to what they would produce (cf. Selden and Selden, in press). These students also tended to view the rigor and precision that are required of a formal proof to be purely academic, or in accordance with a mathematical ritual (cf. Harel and Sowder, 1998). Throughout the course, there were proofs in which the order of the formal presentation differed significantly from the process of creating the proof. "Delta-epsilon" proofs and the verification of identities via induction are two common examples. The undergraduates who viewed proof as process could not comprehend why one should present a formal argument in a different order than by which it was produced; they were genuinely baffled by what they perceived to be a bizarre mathematical convention.

As Simpson (1995) suggests, there may be multiple ways that one can learn the concept of proof. In this paper, I have described one learning trajectory that students in a real analysis course followed to make sense of proof techniques. When a student first learns to construct proofs by following a set of mechanical steps, it is tempting to minimize that student's efforts as "rote learning". Hence it is important to note that students like David can use a procedural understanding of a proof technique as a basis for developing a sophisticated understanding of the concept of proof. However, it is equally important to note that most students who attempted to learn proof procedurally were unable to view a proof as an argument by the end of the course. In some respects, undergraduates who were only able to understand proof as a process performed adequately in real analysis; they were able to construct a wide variety of proofs and some went on to earn a high grade in the course. However, these undergraduates' limited view of proof also caused much confusion about how to formally present a mathematical argument, left them unable to effectively validate their own work, and lead them to acquire misleading beliefs about advanced mathematics.

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